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GeoGebra As a Tool to Enhance Understanding of the Concept of Derivative of a Function and Develop Mathematical Competencies

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ABSTRACT: *Derivative has a central role in calculus. Most students have conceptual difficulties regarding derivative in terms of understanding and giving sense to it. This study conducted a teaching experiment, supported by the use of GeoGebra software to design mathematical tasks on the derivative of a function at a point for an 11th grade classroom. This class of 32 students was divided into eight groups of four students, and each group had the opportunity to operate with the model on GeoGebra. The results showed that the numerical representation, the graphical representation, and the algebraic representation help students understand the concept of derivative. Through observing students' actions as they carried out tasks, we are able to assess the mathematical competencies that students have achieved.*

KEYWORDS: the concept of derivative, GeoGebra software, representations of mathematical competences.

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1. Introduction

Derivative is an important concept of calculus. It is the core knowledge related to many other kinds of knowledge including monotony, extreme value, maximum value, minimum value. Therefore, it is important to ensure that students have a clear understanding of this concept. However, the concept of derivative is epistemologically difficult for students. Most students have conceptual difficulties regarding derivative in terms of understanding and giving sense to it.

The Vietnamese Mathematics Curriculum 2018 has identified five mathematical competencies needed in learning mathematics, including mathematical thinking and reasoning competency; mathematical modelling competency; mathematical problem solving competency; mathematical communication competency; and the use of aids and tools in learning mathematics. Indeed, the question how to teach students to form and develop mathematical competence is a concern of teachers and educators worldwide.

GeoGebra is an application of technology that provides an opportunity for learners to visualize their ideas using graphic illustrations. Using Geogebra promotes students' meaningful and conceptual understanding of the concept of derivative. The prominent feature of GeoGebra is how it combines the properties of both CAS and DGE in a single software package (Hohenwarter & Fuchs, 2004). This software allows users to see the algebraic, graphical and spreadsheet forms of any mathematical objects at the same time (Hohenwarter & Jones, 2007). Using math software like GeoGebra is the current trend in teaching mathematics. Therefore, it is necessary to research the teaching design with the support of GeoGebra software to form and develop students' mathematical competences. This papere thus focuses on answering the following research questions:

Research question 1. What representations can be created to understand the concept of derivative of a function with the assistance of GeoGebra software?

Research question 2. How do these representations help students develop mathematical competence?

2. Research framework

2.1. Defining 'understanding'

Hiebert and Carpenter (1992) specifically defined mathematical understanding as involving the building up of the conceptual ‘context’ or ‘structure’ mentioned above. The authors stated that *“The mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of its connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections”* (p.67). Barmby, Harries, Higgins & Suggate (2007) further defined mathematical understanding as *“to understand mathematics is to make connections between mental representations of a mathematical concept. Understanding is the resulting network of representations associated with that mathematical concept”* (p.2). These definitions draw together an idea of understanding being a network of internalised concepts with the clarification of understanding as an action and a result of an action.

2.2. Three phases of learning a concept

According to Kant, *“all human cognition begins with observations, proceeds from thence to conceptions, and ends with ideas”* (Polya, 1965, p.103). In this sentence, the terms observation, conception, and idea were used. Polya re-expressed the sentence as: *“learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits”* (Polya, 1965, p.103). This suggests that there are three phases in learning a concept, including (1) exploration, (2) formalization, and (3) assimilation. The first phase of exploratory is closer to action and perception and moves on to a more intuitive and heuristic level. The second phase of formalisation ascends to a more conceptual level, introducing terminology, definitions, and proofs. The phase of assimilation comes last, claiming that there should be an attempt to perceive the “inner ground” of things; the material learned should be mentally digested, absorbed into the system of knowledge, into the whole mental outlook of the learner. This phase paves the way to applications on the one hand, and to higher generalizations on the other. As Polya stated, *“for efficient learning, an exploratory phase should precede the phase of verbalization and concept formation and, eventually, the material learned should be merged in, and contribute to, the integral mental attitude of the learner”* (1965, p.104).

2.3. Representations and its role in teaching and learning

According to Davis (1984), *“any mathematical concept, or technique, or strategy – or anything else mathematical that involves either information or some means of processing information – if it is to be present in the mind at all, must be represented in some way”* (p.203). In other words, a representation is defined as any configuration of characters, images, or concrete objects, which can symbolize or “represent” something else (Goldin, 1998). Tadao (2007) classifies representations in math education into five more specific forms as following:

I. Realistic representation: Representations based on the actual state of the object. This type of representation can be directly, specific and natural effects.

II. Manipulative representation: they are teaching aids tools, replacement or imitation of objects that students can affect directly. This type of representation can be very specific and artificial.

III. Visual representation: Representation using illustrations, diagrams, graphs, charts. This is a kind of visual and lively representation.

IV. Language representation: These representations use pure language to express (say or write). This type of representation is governed by conventions, but lacks in succinctness; On the other hand, this representation is descriptive and can create a sense of familiarity.

V. Represented by algebraic symbols: Representations using mathematical symbols such as numbers, letters, symbols.

Mathematical knowledge is usually capable of processing different representations such as symbolic, visual or verbal. According Tadao (2007), different representations have different advantages for displaying and manipulating information. Using dynamic mathematical models with multiple representation systems shown on the same screen, students can switch among them while investigating a specific concept, and can conduct dynamic manipulations on one representation while observing the influences to others. Studies by Finzer and Jakiw (1998) as well as Goldin and Shteingold (2001) about the systems of representations showed that effective mathematical thinking involves understanding the relationships among different representations of "the same" concept as well as the structural similarities (and differences) among representational systems. Duval (2002) further claimed that the conversion of a mathematical concept from one representation to another is a presupposition for successful problem solving.

Mathematical representation is at the heart of understanding a mathematical concept and problem-solving activity of a person. The flexible use of representations could be more meaningful and more effective in studying mathematics. Therefore, teachers need to use multiple mathematical representations to enhance students' understanding of mathematical concepts. According to Vui (2009), *"multiple modes of representation improve transitions from concrete manipulation to abstract thinking, and provide a foundation for continued learning"* (p.2). Visual representation plays an extremely important role in mathematics education, since it allows students to *"think through what is represented (as a method of thinking), record what was thought through the representations (as a method of recording), and thus being an important method for communication"* (p.3). Arcavi (2003) further proposed that *"visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings"* (p. 217).

According to Nam and Stephens (2014), *"the concept of calculus such as the limit of a sequence, the limit of a function, derivative are difficult to teach and understand"*. To overcome this barrier, it is important to generate activities for students to understand the concept intuitively. This research, therefore, design mathematical tasks to help students recognize the idea of derivative intuitively with the support of the GeoGebra software. GeoGebra is not only designed for teachers but also for students so that they can easily focus on mathematical tasks via the technology. GeoGebra supports three external representations of function, including the numerical representation (through tables of values), the graphical representation, and the algebraic representation to help students understand the concept of derivative.

3. Methodology

Classroom-based research which explores the real power of technology for education requires innovative approaches that overcome the traditional class comparison (Duval, 2002). Under the emergent paradigm of design research (Confrey, 2006), a teaching experiment was conducted by a research team including two researchers in mathematics education and a secondary mathematics teacher (García, 2011). In order to gather data to answer the specific research questions in this paper, the author designed a mathematical task on the derivative of a function at a point for a class of 11^{th} graders. This class of 32 students was divided into eight groups of four students. When one group operated the model using the computer, other groups were asked to observe and comment on the images on the screen. Their results were also provided to the other groups to complete their tasks. Evenly, each group had the opportunity to operate with the model, and even re-implemented activities that the previous groups had completed. This process of observation helped later groups gain more experiences, and avoid mistakes from the previous groups as well as to create new ideas. While working with the dynamic model, each group appointed two members, with the first conducting manipulations and the second recording the results. The other two groups were able to suggest solutions. Handouts from all groups were collected for data analysis.

The concept of the derivative of a function is difficult to teach and understand and there is no simple way to help students understand the definition all at once. Our teaching concept design is that first, creating activities for students to understand the concept more or less intuitively. Then, conducting activities to help students gradually refine a more precise definition, and to comprehend the concept's mechanical and geometrical meanings. To help students access the concepts and facilitate the creation of knowledge, four approaches were used, namely a numerical approach, a graphical approach, a verbal approach, and an algebraic approach. In the implementation of these activities, students obtain certain ideas related to the concept. Therefore, open-ended questions were set to provide students the opportunity to propose innovative ideas.

The first designed task was to help students recognize that as $\Delta x = x - x_0$ becomes smaller, the ratio $\dfrac{f(x)-f(x_0)}{x-x_0}$ more accurately reflects the movement's speed, and from that, a basis for the emergence of the derivative concept of a function at a point will be formed. To achieve this goal, it is necessary to go through two levels corresponding to the results of RQ1 and RQ2. First, learning through the manipulation of models and table of corresponding values between Δx and the previous value given Δx, then students must complete the given table with three smaller values of Δx. This is to create the belief that the smaller Δx becomes, the more the ratio $\dfrac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to the given number. The visual image is intended to serve as the creation of a premise for students to have correct confirmation in RQ2. It is important for the learning process that students can comprehend the process of concept formulation, and through it, to form knowledge about methods and develop new ideas.

The design of mathematical tasks

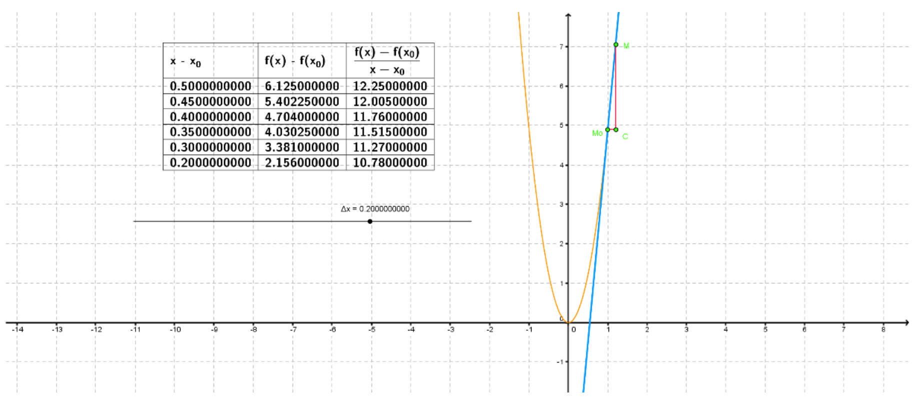
According to Nam and Stephens (2014), it is important to choose appropriate mathematical tasks and activities for students. First, tasks focusing on the concept of derivative of a function at a point should be designed to actively engage students in mathematical thinking. Second, these tasks need to take into account students' previous knowledge and experiences. The task needs to have the following three characteristics:

Introduction to the task. Students see what the model contains. All of its elements and objects are described in sufficient details by the teacher who shows students how to interact with the objects.

Free investigations. The researchers suggest simple manipulations that students can do with models. The amount of time for this action is limited to 5 minutes.

Task questions. The main part of the lesson requires students to conduct several investigations outlined below, and to answer questions. The level of difficulty of the questions increased gradually from visual to abstract, and from particular cases to the general.

Specifically, in the introduction, students releases an object that will freely fall to the ground in order to study its motion. By selecting the vertical axis Oy, the origin O as an initial position



$x - x_0$	$f(x) - f(x_0)$	$\dfrac{f(x)-f(x_0)}{x-x_0}$
0.5000000000	6.125000000	12.25000000
0.4500000000	5.402250000	12.00500000
0.4000000000	4.704000000	11.76000000
0.3500000000	4.030250000	11.51500000
0.3000000000	3.381000000	11.27000000
0.2000000000	2.156000000	10.78000000

Figure 1. Graph of function $f(x) = 4.9x^2$.

of the ball (at time $x = 0$). Ignoring air resistance, the ball's motion equation is determined as $y=\frac{1}{2}gx^2, g=9.8.$ Figure 1 shows the graph of the function $f(x)=4.9x^2.$ The students are asked t0 take a survey with the model in following manner: Change the value $\Delta x = x - x_0$ by dragging the black dot on the Δx slider to the left or right. They are then asked to observe the changes of $f(x)-f(x_0), \frac{f(x)-f(x_0)}{x-x_0}$ and graph of given function.

In question 1, students are asked to change the value of Δx by dragging the black dot on the Δx slider to the left or to the right and complete the calculation sheet of average speed $v_{tb}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ of the time interval Δx from the time moment $x_0 = 1$ second. The value of Δx should be filled in the blank squares so that the following value is smaller than the previous one and smaller than 0.0001. What happens to the value of the average speed as Δx gets smaller and smaller?

The question 2 explores the quantity shows the most accurate speed of motion at the moment of time, x_0.

4. Research results

4.1. Understanding the concept of derivative through representations

Models designed with GeoGebra supports three representations of derivative: the numerical representation; the graphical representation, and the algebraic representation. The numerical representation is represented through corresponding values between and $f(x)-f(x_0), \frac{f(x)-f(x_0)}{x-x_0}$. The graphical representation is shown through a function graph, changing the value $\Delta x = x - x_0$ by dragging the black dot on the slider to the left or right then changing the value of delta x on the model then the segments representing the absolute value of will change. The algebraic representation expresses through expressions and symbols. The representations on GeoGebra provide an opportunity for students to understand the derivative through the following four approaches: a numerical approach, a graphical approach, a verbal approach, and an algebraic approach.

Via numerical approach, students used the formula $v_{tb}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, then calculated the values to complete the table. The first task is designed to help students recognize that as $\Delta x = x - x_0$ becomes smaller, the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ more accurately reflects the movement's speed, and from that will make a basis for the emergence of the concept of the derivative of a function at a point.

Via graphical approach, through the manipulation of models and table of corresponding values between Δx and the previous value given Δx, the students must complete the given table with three smaller values of Δx. This is to create the conclusion that the smaller Δx becomes, the more the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to a given number. The resulting visual image is intended to serve as the creation of a premise for students to have correct confirmation in Question 2.

Table 1. Corresponding values between Δx and v_{tb}

Δx	0,01	0,001	0,0001			
$v_{tb}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$						

Via verbal approach, students move x closer to 1 and observe the table from which to realize that, when x moves closer to 1 or values of Δx gradually approach 0, then values of ratio $\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to 9.8 or $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ = 9.8. This result facilitates students to approach language in formal definition.

Via algebraic approach, expressing values of Δx gradually tends to 0, then values of ratio $\frac{f(x)-f(x_0)}{x-x_0}$ gradually tends to 9.8. This helps students develop an understanding of an algebraic representation, that is, $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ = 9.8. This result makes it convenient for students to generalize the formula into a formal definition.

4.2. Develop mathematical competencies

Through the implementation of learning tasks, students have the opportunity to form and develop mathematical competencies. Firstly, the first competence named *mathematical thinking and reasoning competency,* or being able to competently observe, especially detect the differences in rather complicated situations. This competence helps students recognize that as becomes smaller then the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to a given number and explain the result of the observation.

The second competency, *mathematical modeling competency*, refers to when students establish the mathematical model $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ to describe the situation. This formular is the most accurate speed of motion at the moment of time, x_0.

The third competency, *mathematical problem solving competency*, refers to the identification of the problem in RQ1, that is to find the quantity shows the most accurate speed of motion at the moment of time, x_0. Students with this competencyshould be able to selecting and establish the methods, process of solving the problem based on (1) the table of corresponding values between Δx and v_{tb}, and (2) the definition of the limit of the function at a point to determine the quantity shows the most accurate speed of motion at the moment of time, x_0, that is $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$.

The fourth competency, *mathematical communication competency* refers to how students are able to use the mathematical language in logical combination with a common language to express the thinking, reasoning, proving a mathematical statement, such as, the term “tends to”, “gets smaller and smaller”, “$\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to 9.8” to $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ = 9.8.

The last competence, *competency to using tools and means of learning mathematics*, allows students to identify the names, uses, rules of using on model of the Δx slider; describe how to use mathematical tools and means to perform mathematical tasks by dragging the black dot on the Δx slider to the left then the value of x will become smaller, and to the right then the value of x will become greater.

In the following table, the eight student groups ae numbered G.1, G.2, G.3, G.4, G.5, G.6, G.7, G.8. Performed operations are marked with “v”, non-executed operations are marked with “x”. The results showed that manipulation on the software has helped groups perform operations smoothly, some operations that groups did not perform are often in operations that require students to reason or language switching.

5. Discussion

In cases where some students have difficulties in performing learning tasks, teachers need to support students to overcome learning difficulties. For Task 1, during the experimental process to answer Question 1, there were two groups that

Table 2. Checklist of students' mathematical competence

	G.1	G.2	G.3	G.4	G.5	G.6	G.7	G.8
Recognizing that as becomes smaller then the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to a given number but not being able to explain the result of the observation.	v	v	v	v	v	v	v	v
Recognizing that as becomes smaller then the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to a given number and being able to explain the result of the observation.	v	x	x	v	v	v	x	x
Establishing the mathematical model $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ to describe the situation: the most accurate speed of motion at the moment of time, x_0	x	x	x	v	x	v	v	x
Identifying the problem in RQ1, that is to find the quantity shows the most accurate speed of motion at the moment of time, x_0	v	v	v	v	v	v	v	v
Selecting and establishing the methods, process of solving the problem based on the table of corresponding values between Δx and v_{tb};	v	v	v	v	v	v	v	v
Based on the definition of the limit of the function at a point to determine the quantity shows the most accurate speed of motion at the moment of time, x_0, that is $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.	v	x	v	v	x	x	x	v
Being able to use the mathematical language in logical combination with a common language to express the thinking, reasoning, proving a mathematical statement, such as, the term "tends to", "gets smaller and smaller", "$\frac{f(x)-f(x_0)}{x-x_0}$ gradually comes closer to 9.8" to $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ = 9.8.	v	v	x	v	v	v	x	v
Identifying the names, uses, rules of using on model of the Δx slider; describing how to use mathematical tools and means to perform mathematical tasks:	v	v	v	v	v	v	v	v
Dragging the black dot on the Δx slider to the left then the value of x will become smaller, and to the right then the value of x will become greater.	v	v	v	v	v	v	v	v

did not give the wanted results. The researchers then provided further instruction, "please move the slider to the left to decrease the value of Δx and look at the change of the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ and functional graph". For Question 2, some groups faced difficulty in answering. The researchers then supported by asking students to move slowly and observe the table to see the results.

Students still make mistakes when performing learning tasks, in such situation teachers need to support and should not give correct answers to students. For Question 2, Task 1, one group stated that "no value express accurately motion speed at moment x_0". In this case, the researchers asked them to move x closer to 1 and observe the table from which to realize that, when x moves closer to 1 or values of Δx gradually fade smaller and smaller, then values of v_{tb} gradually tend to a given number. This suggests that students can understand intuitively that $\lim\limits_{x \to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ is the value most accurately expressing the motion's speed at moment x_0.

In conclusion, Dynamic Geometry Systems (in particular, GeoGebra) influence students' learning. The work with DGS made a significant contribution to students' development of mathematical competencies (Albaladejo, García & Codina, 2015). DGS turned out to be a powerful resource for the development of those competencies related to visualization processes, that includes, U*se of Aids and Tools,* together with *Representation*. The software, on the other hand, by means of its attributes and properties, made an important contribution to the development of those competencies related to mathematization, namely, modelling and problem solving.

6. Conclusion

The application of technology as an alternative to teaching and learning is very important. GeoGebra has become a powerful tool that help teachers design effective instructional lessons that enhance teaching and learning process. Dynamic manipulations have potentials to create a new approach in teaching and learning mathematics in school. Using of dynamic models help students construct knowledge of derivative more easily. The building of a teaching model of the concept of derivative with the support of GeoGebra software has really created different representations of the concept of derivative: the numerical representation; the graphical representation, and the algebraic representation. From there, it creates opportunities for students to gain understanding from four approaches: a numerical approach, a graphical approach, a language approach, and an algebraic approach. The task with the model is closer to action and perception and moves to a more intuitive, experiential level, thereby facilitating the formalization phase, that is, moving up to the more conceptual, introductory level. Moreover, the ability to establish meaningful links between representational forms and the concept being represented is at the heart of what it means to understand the concept of derivative of a function at a point.

The designed learning tasks represent the three stages of concept learning: (1) unconscious play in concrete situations (construction), (2) following by the realization of something meaningful among the plays (transition from construction to analysis), and (3) ending with a moment of insight and understanding into the meaningfulness (analysis). GeoGebra supports and enhances learners' understanding of derivative, and trains learners' reasoning skills so students can become problem solvers and math communicators. Through specific operations when performing the designed tasks, this teaching model helps students form and develop mathematical competencies specified in the Vietnamese mathematics Curriculum 2018: mathematical thinking and reasoning competency; mathematical modeling competency; mathematical problem solving competency; mathematical communication competency; and competency to using tools and means of learning mathematics.

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